**Interaction between multiple piezoelectric inclusions and a crack in a non-piezoelectric matrix**

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**Abstract:**

The current study investigates the interaction between multiple fiber-shaped piezoelectric sensors (the inhomogeneity or inclusions) and a nearby crack. The piezoelectric sensors are embedded in a non-piezoelectric elastic matrix which contains a crack. The matrix is subjected to a far field in-plane tension and the fibers are applied by anti-plane electric loading at infinity. The interaction between the inclusions and the crack is partially decoupled through a decomposition process. During the solution procedure, the crack is simulated by a continuous distribution of edge dislocations. By employing the solution of an edge dislocation interaction with multiple inclusions in elastic media as the Green function, the problem is formulated into a set of singular integral equations, which are solved by numerical method. The stress intensity factors (SIFs) are derived in terms of the asymptotic values of the dislocation density functions evaluated from the integral equations. The numerical results indicate that the electric loading plays an important role in the interaction between multiple piezoelectric inclusions and the crack, the influence of the ‘softer’ piezoelectric inclusions on the crack is quite different from the ‘harder’ piezoelectric inclusions, the geometric parameters have a remarkable influence on a crack, the SIFs is greatly affected by the number and array type of inclusions.

Keywords: A. Fibers; A. Smart materials; B. Electrical properties; B. Mechanical properties; C. Crack;

**1. Introduction**

Piezoelectric materials are widely used in more and more sophisticated engineering fields because of their intrinsic electromechanical coupling behavior. Piezoelectric ceramics based on ferroelectric crystals such as lead zirconate titanate (PZT) and barium titanate (BaTiO3) are employed as electromechanical sensors, transducers and actuators [1] [2]. A lot of investigation has been done to the practical importance of piezoelectric materials, such as Deeg [3], Pak [4], Suo et al. [5] , etc. Inclusions (or Inhomogeneities) either unwanted or deliberately introduced in the materials can dramatically change their mechanical properties. Therefore the study of inclusions has received a considerable amount of attention (see, for example [6] [7] [8] [9]). When multiple piezoelectric fibers are used as sensors, they are usually embedded in non-piezoelectric matrix with a crack. It is desirable to understand the interaction between multiple piezoelectric sensors and the near-by defects in the matrix.

The interaction problem between cracks and inclusions in materials has been an important topic in the literature. For instance about traditional crack-inclusion interaction problems in pure elastic media, the interaction between a crack and a circular inclusion in a sheet under tension was studied by Tamate [10] and an exact solution of the stress field at the neighborhood of the inclusion was obtained. Atkinson [11] first studied the interaction problem between a crack and an inclusion by using numerical method. Erdogan et al.[12] considered the interaction between an isolated circular inclusion and a line crack embedded in an infinite matrix with the distributed dislocation method. The interaction between an elastic circular inclusion and two symmetrically placed collinear cracks was investigated by Hsu and Shivakumar [13]. The distributed dislocation method is an effective tool to solve various kinds of crack problems [14] [15] [16] [17] [18] [19] . The investigation for a crack near an elliptic inclusion was carried out in terms of the body force method by Nisitanietal[20]. Luo and Chen [21] investigated the matrix cracking in fiber-reinforced composite materials. Liu etal. [22] studied the effects of imperfect bonding on stress intensity factors calculated at a radial matrix crack in a fiber composite subjected to various cases of mechanical loading. Xiao and Chen [23] studied the interaction between a radial matrix crack and a three-phase circular inclusion. Kim and Sudak [24] investigated the interaction between a radial matrix crack and a three-phase circular inclusion with imperfect interface in plane elasticity . Patton and Santare [25] investigated the effect of a rigid elliptical inclusion on a straight crack. As for such problems in piezoelectric materials, Sosa [26] presented a two-dimensional electroelastic analysis in piezoelectric media with defects. Dunn and Wienecke [27] analyzed the electroelastic field in and around inclusion and inhomogeneities in piezoelectric solids. Qin [28] obtained the thermoelectroelastic solution for an elliptic piezoelectric inclusion embedded in an infinite matrix and applied the result to solve crack-inclusion problems. Xiao and Bai [29] derived the stress field and the stress intensity factor for a Griffith crack located near a piezoelectric inclusion in an infinite non-piezoelectric matrix. However, most of the analytical solutions presented in the literature are restricted to the problem involved in the interaction of a single piezoelectric inclusion and a crack. In fact, multiple piezoelectric inclusions are usually embedded in a non-piezoelectric matrix with a crack, and moreover the interactions between a crack and multiple inclusions have to be considered when they are closely arrayed.

**2. Physical problem and decomposition process**

In a rectangular coordinate system (j=1,2,3) , we consider an infinite elastic matrix containing N cylindrical piezoelectric inclusions which are parallel to each other along the  direction . A crack with length 2c contained in the matrix locates at the origin of the coordinate. It is assumed that the matrix is isotropic, while the inclusions are transversely isotropic and polarized along the symmetry axis. The matrix is subjected to a far field in-plane uniform tension  and the inclusions are loaded by a uniform electric field  in the  direction. The crack line is along the -axis, and thus perpendicular to the far field tensile stress. Additionally, assume that all the inclusions are completely bounded to the matrix. The cross-section of the system is shown in Fig.1, where the regions occupied by the matrix and the inclusions are denoted by ‘m’ and ‘f’, respectively, and  represents the radius of any inclusions.

Fig.1. the interaction between multiple circular piezoelectric inclusions and a crack in an infinite elastic matrix.

As the matrix is pure elastic material, there is no mechanical-electric coupling behavior inside the matrix. By employing the superposition principle of elasticity [30], the solution of the present problem can be obtained through the sum of two sub-problems, as shown in Fig. 1. The sub-problem I shown in Fig. 2 is the piezoelectric inclusions embedded in the matrix without the crack. For the sub-problem II shown in Fig. 3, the only external loads are the crack surface tractions which are equal in magnitude and opposite in sign to the stresses obtained in the first problem along the line which is the presumed location of the crack. The superposition of sub-problem I and sub-problem II is thus equal to the original problem.

Fig.2.sub-problem I

Fig.3.sub-problem II

**3. Solution scheme**

The sub-problem I has been solved in the ref. [31]. In the matrix (an infinite plane with N circular holes), the complex potentials can be written in the form of power series

 (1)

where  and  are unknown coefficients,  is the centre point of the rth inclusion,  stands for a reference length which may be defined as , and are related to the stress state at infinity:

 (2)

On the other hand, the complex potentials inside the inclusions  can be expressed as

 (3)

where  and  are unknown coefficients. The unknown coefficients can be determined [31], and then the stress fields of sub-problem I are related to the complex variables through

 (4)

In sub-problem II, the crack can be simulated by an array of edge dislocations with unknown densities and  (the subscripts andrepresent different components of the edge dislocations) based on Bueckner’s theorem [32]. The solution of an edge dislocation interaction with N piezoelectric inclusion in elastic media should be obtained first as the Green function.

Fig.4. an edge dislocation interacting with N piezoelectric inclusions in elastic media

As shown in Fig.4, a single edge dislocation with Burger’s vector  locates at the point. The problem can be solve using the similar method in the ref.[31]. In the matrix, the complex potentials have the form of

 (5)

where  and  are unknown coefficients, .On the other hand, the complex potentials inside the inclusions  can be expressed as

 (6)

where  and  are unknown coefficients. To satisfy the boundary condition, the unknown coefficients should obey the following equations:

 (7)

where









Eqs. (7) and their conjugated equations constitute a set of  linear equations concerning  unknown coefficients . After they are determined, all the complex potentials about the matrix and inclusions are known, and then the stress fields produced by the dislocation located at can be expressed as

 (8)

Substituting into Eq. (8) , the shear stress  and the normal stress at the point produced by a unit glide dislocation at  can be given in the form of

 (9)

where ,represent the regular part of the fundamental solution of the unit glide edge dislocation interacting with the N circular inclusions.

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The stress field of sub-problem II can be obtained through an integral over the fundamental solutions along the crack line.

 (11)

where ,  and are the dislocation density components at the point .The single-value condition of displacement vector requires that the density functions of the system satisfy the following relation:

 (12)

The traction free boundary condition requires that the normal and shear stress components along the crack surface are zero, i.e.

 (13)

Substituting Eqs. (9), (10) and (11) into Eqs. (13) , they can be rewrite as



(14)

Let , we can transform Eqs.(14 ) into two simple Cauchy-type singular integral equations as follows:



(15)

The additional condition (12) is rewritten as:

 (16)

As the crack is embedded in a homogeneous isotropic matrix, both crack tips have the square root singularities. According to Erdogan [33], it is apparent that Eqs. (15 ) are two singular integral equations with index +1, and the fundamental solution of the singular equations is:

 (17)

where and  are bounded functions in the interval . According to the Cause-Chebyshev solution [33], Eqs. ( 15) and Eqs. (16 ) are discreted, and we get a group of  linear algebra equations with the  unknown .



(18)

Where



Once the dislocation density functionare evaluated, following Edorgan [12], the stress intensity factors (SIFs) at the both crack tips can be expressed as:

 (19)

where  is the right crack tip and  is the left crack tip.

They can be normalized by, then

 (20)

**4. Results and discussion**

Chose the PZT-5H as a model material of inclusions, the value of material constants for PZT-5H are listed in Table1[4]. We assume that Poisson’s ratio of the matrix is.

Table1. elastic coefficients, piezoelectric coefficients, dielectric coefficients

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | *C*11 | *C*33 | *C*44 | *C*12 | *C*13 | *e*13 | *e*33 | *e*15 | **11 | **33 |
| PZT-5H | 12.6 | 11.7 | 3.53 | 5.5 | 5.3 | -6.5 | 23.3 | 17.0 | 151 | 130 |



Fig.5 the SIFs for a crack interacting with a single piezoelectric inclusion, where Xiao’s solution is found in the ref.[29].

To verify the rightness of the present results, we calculate the variation of  at the right tip of the crack located near a single piezoelectric inclusion with the distance factor , where  is the radius of the piezoelectric inclusion and  is the distance between the centers of the crack and the inclusion. The results shown in Fig.5 can be compared with the solution provided by Xiao and Bai [29]. It is found that the present solution is well agreeable to Xiao’s solution.



Fig.6 the SIFs for a crack located at the center of two same piezoelectric inclusions, change the position of the two inclusions synchronously.

That shown in Fig.6 is the variation of the SIFs for a crack interacting with two same piezoelectric inclusions with the distance factor. In this case, we change the position of the two piezoelectric inclusions synchronously. Compared with a single piezoelectric inclusion, two ‘softer’ piezoelectric inclusions (having lower yang’s model than the matrix) increase the SIFs while two harder piezoelectric inclusions decrease it. The effect of the inclusions will be getting weaker as the distance between the crack and the inclusions increase.



Fig.7 variation of the SIFs for a crack interacting with two same piezoelectric inclusions with the electric loading.

To investigate the influence of the electric loading on the crack, we calculate the variation of  for a crack located at the center of the two same piezoelectric inclusions with the normalized electric loading. Due to the electro-mechanical coupling effect of the piezoelectric material, the influence of the electric loading is significant, especially for the ‘softer’ inclusions. It is found from the Fig.7 that the influence on the SIFs is linear, the SIFs decreases as the electric loading increases. It indicates that only when the electric filed increases in a certain direction does the SIFs increase. The SIFs tends to be infinitely large whenincrease along the negative direction of . Hence, the crack is very dangerous when ranges in the negative value. Compared with a single piezoelectric inclusion, two ‘softer’ piezoelectric inclusions make the SIFs decrease more rapidly while the change can be neglected for two ‘harder’ piezoelectric inclusions.



Fig.8 the SIFs at the right tip of a crack interacting with two different piezoelectric inclusions.

That shown in Fig.8 is the variation of at the right tip of a crack interacting with two different piezoelectric inclusions. In this case, we guarantee the distance between the center of the crack and the right edge of the left inclusion is equal to that between the center of the crack and the left edge of the right inclusion. It is found that for the ‘softer’ inclusions, the SIFs increases as the radius of the right inclusion increases, but for the ‘harder’ inclusions, it decreases as the radius increases. It indicates that a bigger piezoelectric inclusion can reduces the SIFs or increase it, depending on the relative stiffness of the inclusions and the matrix. When the ‘softer’ piezoelectric inclusion is replaced by a bigger one or a ‘harder’ piezoelectric inclusion is replaced by a smaller one, the crack will become more dangerous.



Fig.9 the SIFs at the both tip of the crack interacting with two same piezoelectric inclusions, fix the left inclusion and change the position of the right one.

That shown in Fig.9 is the variation of  at both tips of a crack interacting with two same piezoelectric inclusions with the distance factor. In this case, the left inclusion is fixed and the position of the right inclusion can be changed. For the ‘softer’ inclusions, when the right inclusion is closer to the crack than the left (), the SIFs at the right tip is larger than that at the left tip; when the right inclusion moves far away (), the SIFs at the right tip is smaller than that at the left tip; it indicate that the tip which is closer to the inclusions is more dangerous than the other. On the contrary, the tip which is closer to the inclusions is much safer than the other for the ‘harder’ piezoelectric inclusions.



Fig.10 the SIFs at the right tip of the crack interacting with the regular array I of three piezoelectric inclusions

That shown in Fig.10 is the variation of  at the right tips of the crack interacting with the regular triangular array of three same piezoelectric inclusions with the distance factor, where is the distance between the center of the crack and the right inclusion. In this case, the regular triangular array of three piezoelectric inclusions is symmetrical about. Under the electric loading, for the ‘harder’ piezoelectric inclusions, the value of the SIFs tends to the maximum when the crack locates between the two left inclusions; on the contrary, for the ‘softer’ inclusions, the value of SIFs is negative, so the crack is closed and safe. Compared with the electric loading, it is worth notice that the influence of the electric loading on the crack is very significant for the ‘softer’ inclusions while it is very weak for the ‘harder’ inclusions.



Fig.11 the model II SIFs at the right tip of the crack interacting with the regular triangular array II of three piezoelectric inclusions



Fig.12 the model I SIFs at the right tip of the crack interacting with the regular triangular array II of three piezoelectric inclusions

If the regular triangular array of three piezoelectric inclusions is not symmetrical about, the model II SIFs for the crack is not equal to 0. The variation of  at the right tips of the crack is shown in Fig.11 and the variation of  is shown in Fig.12. Under the electric loading, for the ‘softer’ piezoelectric inclusions, the value of  slowly increases to the maximum, then rapidly decreases to the minimum as the crack moving to the inclusions from far away. When the crack locates in the inclusions, the value of the model I SIFs approaches to zero but the value of the model II is very large; for the ‘harder’ piezoelectric inclusions, when the crack locates in the inclusions, the model I SIFs and the model II SIFs are all in high level. Compared with the electric loading, the ‘softer’ inclusions make the model I SIFs decrease greatly and the amplitude of the model II SIFs increase significantly but there is hardly any change for the ‘harder’ inclusions.

**5. Conclusion**

The interaction problem of multiple piezoelectric inclusions embedded in an infinite non-piezoelectric matrix which contains a crack is investigated. Based on the superposition principle of elasticity, the problem can be obtained through the sum of two sub-problems. The sub-problem I has been solved. In sub-problem II, the crack is simulated by a continuous distribution of edge dislocations with unknown density. The solution of an edge dislocation interaction with multiple inclusions in elastic is derived as the Green functions. The stress field in sub-problem II can be derived through an integral over the Green functions along the crack line. The superposition of the stress field of sub-problem I and sub-problem II should satisfy the traction free boundary condition along the crack surface. Thus a set of singular integral equations are formulated, they can be solved by numerical method. Then the stress intensity factors are derived in terms of the asymptotic values of the dislocation density functions evaluated from the integral equations. Numerical examples are given for various material constants combinations and geometric parameters. The numerical results indicate that

1. The electric loading plays an important role in the interaction between multiple piezoelectric inclusions and the crack. For example, the electric loading along a given direction may cause a crack closed, but applying the opposite electric loading, the crack will be more dangerous.
2. The influence of the ‘softer’ piezoelectric inclusions on the crack is quite different from the ‘harder’ piezoelectric inclusions. The ‘softer’ inclusions are more sensitive to the changes of the geometric parameters and the electric loading. For example, under different electric loading, the changes of the SIFs are great for the ‘softer’ inclusions but there is hardly any change for the ‘harder’ inclusions.
3. The geometric parameters have a remarkable influence on a crack. For example, if one of multiple inclusions is replaced by a bigger one, the SIFs will increase or decrease, depending on the relative stiffness of the inclusions and the matrix.
4. The SIFs is greatly affected by the number and array type of inclusions. When multiple piezoelectric inclusions are interacting with a crack, this method can be used to identify whether the crack is dangerous or not.

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